THE ELASTIC EQUILIBRIUM OF A TRANSVERSELY ISOTROPIC LAYER AND A THICK PLATE

(UPRUGOE RAVNOVESIE TRANSVERSAL'NO IZOTROPNOGO SLOIA I TOLSTOI PLITY)

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The problem of the elastic equilibrium of an infinite isotropic layer is treated in detail in the book by Lur'e [1], who has used a method of solution based on the application of special differential operators. Knowing the solution for an infinite layer, one can readily obtain the solution for a finite thick plate, if the exact boundary conditions on the lateral surface are replaced by approximate ones, integral or averaged.

In this article we consider the problem of equilibrium of an elastic layer which possesses the anisotropy of a particular kind, namely transverse isotropy; the general and the particular solutions are constructed by a method analogous to that of Lur'e.

1. The general equations and formulas for a transversely isotropic medium. Suppose we have an elastic, homogeneous, transversely isotropic body, which follows the generalized Hooke's law and which undergoes small strains under the applied external loads. It is well known that the anisotropy of such a body is characterized by the existence of a plane of isotropy at every point or, which is the same, by the existence of an axis of elastic symmetry of an infinitely high order [2, p.172]. The problem of the elastic equilibrium of a transversely isotropic medium is of definite interest in mining engineering; it is natural to attribute this kind of anisotropy (at least in first approximation) to the sedimentary rocks - sandstones, siltstones, phyllites and others, in which the planes of strata, due to their formation and structure, are the planes of isotropy.

If the z-axis is directed normally to the planes of isotropy and if the common designations for the components of stress and strain are used, the the equations expressing generalized Hooke's law can be written down as follows:

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu_1}{E_1}\sigma_z, & \gamma_{yz} &= \frac{1}{G_1}\tau_{yz} \\ \varepsilon_y &= \frac{1}{E}(-\nu\sigma_x + \sigma_y) - \frac{\nu_1}{E_1}\sigma_z, & \gamma_{xz} &= \frac{1}{G_1}\tau_{xz} \\ \varepsilon_z &= -\frac{\nu_2}{E}(\sigma_x + \sigma_y) + \frac{1}{E_1}\sigma_z, & \gamma_{xy} &= \frac{1}{G}\tau_{xy} \end{aligned} \tag{1.1}$$

Here, E, E_1 are Young's moduli for the extension-contraction in the plane of isotropy and in the transverse direction, ν, ν_1, ν_2 are Poisson's ratios, G, G_1 are the shear moduli for the planes of isotropy and those normal to them. Out of the seven elastic constants only five will be independent, since

$$\nu_2 = \nu_1 \frac{E}{E_1}, \qquad G = \frac{E}{2(1+\nu)} \tag{1.2}$$

Let us introduce the designations

$$\partial_1 = \frac{\partial}{\partial x}, \qquad \partial_2 = \frac{\partial}{\partial y}, \qquad D^2 = \partial_1^2 + \partial_2^2 \tag{1.3}$$

(D^2 is the Laplacian operator for a function of two variables x and y)

$$H = E\nu_1 + G_1(1 - \nu - 2\nu_1\nu_2)$$

$$\alpha = \frac{2GG_1(1-\nu_1\nu_2)}{H}, \qquad \beta = \frac{G_1 E\nu_1}{H}, \qquad \gamma = 2G\frac{1-\nu_1\nu_2}{H}$$

$$\alpha_1 = 2G\frac{E_1 - G_1\nu_1(1+\nu)}{H}, \qquad \beta_1 = G_1E_1\frac{1-\nu}{H}, \qquad \delta = G_1\frac{1-\nu-2\nu_1\nu_2}{H}$$

$$s_0 = \sqrt{\frac{G}{G_1}}, \qquad s_{1,2} = \sqrt{\frac{\alpha_1 - \beta \pm \sqrt{(\alpha_1-\beta)^2 - 4\alpha\beta_1}}{2\beta_1}}$$

$$n = \frac{2}{s_1s_2(\alpha\beta_1 + \alpha_1\beta)} = \frac{H}{s_1s_2E_1G_1G} \tag{1.4}$$

Here, s_1, s_2 are the roots of the equation

$$\beta_1 s^4 + (\beta - \alpha_1)s^2 + \alpha = 0 \tag{1.5}$$

As is well known, the stress components and the projections of the displacement in a transversely isotropic medium, in the general case of deformation, are expressed by means of two functions φ and F, which satisfy the equations [3]

$$\left(\frac{\partial^2}{\partial z^2} + s_0^2D^2\right)\varphi = 0, \qquad \left(\frac{\partial^2}{\partial z^2} + s_1^2D^2\right)\left(\frac{\partial^2}{\partial z^2} + s_2^2D^2\right)F = 0 \tag{1.6}$$

The general expressions for the stresses and displacements have the form

$$\sigma_x = -2G\partial_1\partial_2\varphi + \frac{\partial}{\partial z}\left(2G\partial_2^2 - \alpha D^2 + \beta\frac{\partial^2}{\partial z^2}\right)F, \qquad \sigma_z = \frac{\partial}{\partial z}\left(\alpha_1 D^2 + \beta_1\frac{\partial^2}{\partial z^2}\right)F$$
$$\sigma_y = 2G\partial_1\partial_2\varphi + \frac{\partial}{\partial z}\left(2G\partial_1^2 - \alpha D^2 + \beta\frac{\partial^2}{\partial z^2}\right)F, \tag{1.7}$$

$$\tau_{xy} = G(\partial_1^2 - \partial_2^2)\varphi - \frac{\partial}{\partial z}(2G\partial_1\partial_2 F)$$
$$\tau_{xz} = -G_1\frac{\partial}{\partial z}(\partial_2\varphi) + \partial_1\left(\alpha D^2 - \beta\frac{\partial^2}{\partial z^2}\right)F$$
$$\tau_{yz} = G_1\frac{\partial}{\partial z}(\partial_1\varphi) + \partial_2\left(\alpha D^2 - \beta\frac{\partial^2}{\partial z^2}\right)F \tag{1.8}$$

$$u = -\partial_2\varphi - \frac{\partial}{\partial z}(\partial_1 F), \qquad v = \partial_1\varphi - \frac{\partial}{\partial z}(\partial_2 F), \qquad w = \left(\gamma D^2 + \delta\frac{\partial^2}{\partial z^2}\right)F \tag{1.9}$$

2. The elastic equilibrium of an infinite layer. Consider an infinite, elastic, transversely isotropic, layer of constant thickness h, whose top and bottom planes are parallel, which is in equilibrium under the forces applied to its boundary surfaces; we shall disregard the body forces. It is assumed that the planes of isotropy at any point are parallel to the middle plane. Taking the latter to be the xy plane, we have the equations of generalized Hooke's law in the form (1.1).

Any loading applied to the surfaces can be broken up into two parts: A) the load symmetrical with respect to the middle plane and B) the skew-symmetrical load. The load of type A causes deformations whose characteristic is that the middle surface remains plane and undergoes extension or contraction; the loads of type B are associated with deformations in which the middle surface undergoes bending in such a manner that its linear elements do not change their length. Following Lur'e [1, p.148] let us call the problem associated with the symmetrical loads the problem of extension-contraction, and the one corresponding to the skew-symmetrical loads - the problem of bending. The solutions of both problems will be constructed by means of functions φ and F in the form of series arranged by the positive powers of z. We shall look for solutions of Equations (1.6) in the form

$$\varphi = \sum_{k=0}^{\infty} \varphi_k(x, y) z^k, \qquad F = \sum_{k=0}^{\infty} F_k(x, y) z^k \tag{2.1}$$

Substituting the Expressions (2.1) into the Equations (1.6) and setting the coefficients of identical powers of z equal to zero, we obtain the recurrence differential equations for the functions φ_k and F_k with different indices.

From these equations, the coefficients of φ_k can be expressed in terms of two arbitrary functions φ_0, φ_0' of the variables x and y, and the coefficients of F_k – in terms of four functions F_1, F_2, F_1', F_2'.

It has been proved that the parameters s_1 and s_2 can be real or complex, but cannot be pure imaginary numbers [4]. If they are different, then the final expressions for φ and F can be written in the following compact form

$$\varphi = \cos s_0 zD \cdot \varphi_0 + \sin s_0 zD \cdot \varphi_0' \tag{2.2}$$
$$F = \cos s_1 zD \cdot F_1 + \cos s_2 zD \cdot F_2 + \sin s_1 zD \cdot F_1' + \sin s_2 zD \cdot F_2'$$

Here $\cos s_k zD$ and $\sin s_k zD$ are the differential operators of the form

$$\cos s_k zD = 1 - \frac{s_k^2 z^2 D^2}{2!} + \frac{s_k^4 z^4 D^4}{4!} - \ldots$$
$$\sin s_k zD = s_k zD - \frac{s_k^3 z^3 D^3}{3!} + \frac{s_k^5 z^5 D^5}{5!} - \ldots \tag{2.3}$$

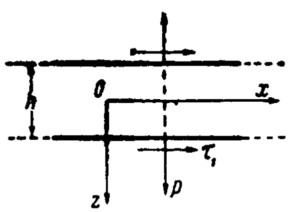

Fig. 1.

From the formulas (1.7) to (1.9) we find the expressions for the stresses and displacements, which contain the same operators (2.3). In the problem of extension-contraction the stress distribution will be symmetrical with respect to the xy plane and we can set in advance $\varphi_0' = F_1 = F_2 = 0$; in the bending problem $\varphi_0 = F_1' = F_2' = 0$. From there on we shall always assume that $s_1 \neq s_2$. In the case of equal parameters all the formulas will be somewhat more complicated, but we shall not look into it, since it can be reduced to the case of an isotropic layer, if a new variable $z' = s_1 z$ is introduced. On the other hand, the solution of any particular problem for $s_1 = s_2$ can be obtained from the solution for the case of unequal parameters by transition to the limit.

3. The extension-contraction of a layer. Suppose that the boundary planes are subjected to given normal and tangential forces distributed symmetrically with respect to the middle surface (Fig. 1).

We designate the magnitudes of the external forces by p, τ_1, τ_2; the normal forces will be taken as positive if they cause tension; for the tangential forces we adopt the sign convention which is generally used for shear stresses.

The stresses satisfy the boundary conditions

$$\tau_{xz} = \pm \tau_1, \qquad \tau_{yz} = \pm \tau_2, \qquad \sigma_z = p \quad \text{for} \quad z = \pm \frac{1}{2} h \tag{3.1}$$

Let us introduce the designations

$$A = -D^3\,[s_1\,(\alpha + \beta s_1^2)\cos s_1 zD\cdot F_1' + s_2\,(\alpha + \beta s_2^2)\cos s_2 zD\cdot F_2']$$
$$B = D\,(s_1 \cos s_1 zD\cdot F_1' + s_2 \cos s_2 zD\cdot F_2') \tag{3.2}$$

From the formulas (1.7) to (1.9), on the basis of the expressions (2.2), we obtain

$$\sigma_x = -2G\partial_1\partial_2 \cos s_0 zD\cdot\varphi_0 + A + 2G\partial_2^2 B$$
$$\sigma_y = 2G\partial_1\partial_2 \cos s_0 zD\cdot\varphi_0 + A + 2G\partial_1^2 B \tag{3.3}$$
$$\sigma_z = D^3\,[s_1\,(\alpha_1 - \beta_1 s_1^2)\cos s_1 zD\cdot F_1' + s_2\,(\alpha_1 - \beta_1 s_2^2)\cos s_2 zD\cdot F_2']$$

$$\tau_{xy} = G\,(\partial_1^2 - \partial_2^2)\cos s_0 zD\cdot\varphi_0 - 2G\partial_1\partial_2 B$$
$$\tau_{xz} = G_1 s_0 \partial_2 D \sin s_0 zD\cdot\varphi_0 + \partial_1 D^2\,[(\alpha + \beta s_1^2)\sin s_1 zD\cdot F_1' + (\alpha + \beta s_2^2)\sin s_2 zD\cdot F_2']$$
$$\tau_{yz} = -G_1 s_0 \partial_1 D \sin s_0 zD\cdot\varphi_0 + \partial_2 D^2\,[(\alpha + \beta s_1^2)\sin s_1 zD\cdot F_1' + (\alpha + \beta s_2^2)\sin s_2 zD\cdot F_2'] \tag{3.4}$$

$$u = -\partial_2 \cos s_0 zD\cdot\varphi_0 - \partial_1 B, \qquad v = \partial_1 \cos s_0 zD\cdot\varphi_0 - \partial_2 B$$
$$w = D^2\,[(\gamma - \delta s_1^2)\sin s_1 zD\cdot F_1' + (\gamma - \delta s_2^2)\sin s_2 zD\cdot F_2'] \tag{3.5}$$

Satisfying the conditions (3.1) for φ_2, F_1', F_2' we get the equations

$$G_1 s_0 \partial_2 D \sin\frac{s_0 hD}{2}\cdot\varphi_0 + (\alpha + \beta s_1^2)\,\partial_1 D^2 \sin\frac{s_1 hD}{2}\cdot F_1' + (\alpha + \beta s_2^2)\,\partial_1 D^2 \sin\frac{s_2 hD}{2}\cdot F_2' = \tau_1$$
$$-G_1 s_0 \partial_1 D \sin\frac{s_0 hD}{2}\cdot\varphi_0 + (\alpha + \beta s_1^2)\,\partial_2 D^2 \sin\frac{s_1 hD}{2}\cdot F_1' + (\alpha + \beta s_2^2)\,\partial_2 D^2 \sin\frac{s_2 hD}{2}\cdot F_2' = \tau_2 \tag{3.6}$$
$$s_1\,(\alpha_1 - \beta_1 s_1^2)\,D^3 \cos\frac{s_1 hD}{2}\cdot F_1' + s_2(\alpha_1 - \beta_1 s_2^2)\,D^3 \cos\frac{s_2 hD}{2}\cdot F_2' = p$$

We shall solve these equations in the same way as in the corresponding case of an isotropic layer [1, pp.153-156]. Let us form the operator-determinant of the system (3.7) Q' and the minors Q_{ij}'; we get

$$Q' = \frac{G_1 s_0}{h} D^6 Q \tag{3.7}$$

where

$$Q = D^2 \sin\frac{s_0 hD}{2}\left[(s_1 + s_2)\sin(s_1 - s_2)\frac{hD}{2} + (s_1 - s_2)\sin(s_1 + s_2)\frac{hD}{2}\right] \tag{3.8}$$

We shall not state the expressions for the minors; their form is clear (see the system (3.6)). Furthermore, let us introduce the three stress functions, requiring that they satisfy the equations

$$Q'\chi_1' = \tau_1, \qquad Q'\chi_2' = \tau_2, \qquad Q'\chi'_3 = p \tag{3.9}$$

For that purpose we have to set

$$\begin{aligned} \varphi_0 &= Q_{11}'\chi_1' + Q_{21}'\chi_2' + Q_{31}'\chi_3' \\ F_1' &= Q_{12}'\chi_1' + Q_{22}'\chi_2' + Q_{32}'\chi_3' \\ F_2' &= Q_{13}'\chi_1' + Q_{23}'\chi_2' + Q_{33}'\chi_3' \end{aligned} \tag{3.10}$$

Now, we introduce the new functions

$$\chi_k = \frac{s_0}{n} D^6 \chi_k' \qquad (k = 1, 2, 3) \tag{3.11}$$

These functions satisfy the equations

$$Q\chi_1 = \frac{\tau_1}{G_1}, \qquad Q\chi_2 = \frac{\tau_2}{G_1}, \qquad Q\chi_3 = \frac{p}{G_1} \tag{3.12}$$

Thus, we obtain the final expressions

$$\begin{aligned} \varphi_0 &= \frac{Q}{s_0 D^3 \sin s_0 hD/2} (\partial_2\chi_1 - \partial_1\chi_2) \\ F_1' &= G_1 n \frac{1}{D^2} \sin\frac{s_0 hD}{2}\Big[s_2(\alpha_1 - \beta_1 s_2^2)\cos\frac{s_2 hD}{2}(\partial_1\chi_1 + \partial_2\chi_2) - \\ &\qquad - (\alpha + \beta s_2^2) D \sin\frac{s_2 hD}{2}\chi_3\Big] \\ F_2' &= G_1 n \frac{1}{D^2} \sin\frac{s_0 hD}{2}\Big[- s_1(\alpha_1 - \beta_1 s_1^2)\cos\frac{s_1 hD}{2}(\partial_1\chi_1 + \partial_2\chi_2) + \\ &\qquad + (\alpha + \beta s_1^2) D \sin\frac{s_1 hD}{2}\chi_3\Big] \end{aligned} \tag{3.13}$$

These functions have to be substituted into the formulas (2.3) to (3.5). The expressions for stresses and displacements are thus obtained, but we shall not state them here. Let us only point out that all the differential operators are commutative and hence no complications can arise in the substitution.

4. Particular cases of extension-contraction. All the formulas and equations are considerably simplified in the two particular cases of loading.

Case 1. Only normal forces are acting on the surface ($\tau_1 = \tau_2 = 0$). Since χ_1, χ_2 satisfy the homogeneous equations, they can be set equal to zero. Instead of χ_3 let us introduce a new stress function

$$\chi = D \sin \frac{s_0 hD}{2} \chi_3 \tag{4.1}$$

This function satisfies the equation

$$D\left[(s_1 + s_2) \sin (s_1 - s_2) \frac{hD}{2} + (s_1 - s_2) \sin (s_1 + s_2) \frac{hD}{2}\right] \chi = \frac{p}{G_1} \tag{4.2}$$

We get $\varphi_0 = 0$ (4.3)

$$F_1' = - G_1 n (\alpha + \beta s_2^2) \frac{1}{D^2} \sin \frac{s_2 hD}{2} \chi, \qquad F_2' = G_1 n (\alpha + \beta s_1^2) \frac{1}{D^2} \sin \frac{s_1 hD}{2} \chi$$

Case 2. The normal forces are absent and the tangential forces have a potential, i.e.

$$p = 0, \qquad \tau_1 = \partial_1 \tau, \qquad \tau_2 = \partial_2 \tau \tag{4.4}$$

We introduce a new function χ^*, setting

$$\chi_1 = \frac{\partial_1 \chi^*}{D \sin s_0 hD/2}, \qquad \chi_2 = \frac{\partial_2 \chi^*}{D \sin s_0 hD/2}, \qquad \chi_3 = 0 \tag{4.5}$$

The function χ^* satisfies the Equation (4.2) in which τ is substituted for p.

5. Bending of a layer. Let a skew-symmetric loading with components $\pm q$, t_1, t_2 per unit area, be given on the boundary surfaces of the layer (Fig. 2). In this case we have the boundary conditions

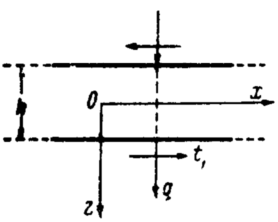

Fig. 2.

$$\tau_{xz} = t_1, \qquad \tau_{yz} = t_2, \qquad \sigma_z = \pm q \qquad \text{for } z = \pm \frac{1}{2} h \tag{5.1}$$

Introducing concise designations

$$A_1 = D^3 [s_1 (\alpha + \beta s_1^2) \sin s_1 zD \cdot F_1 + s_2 (\alpha + \beta s_2^2) \sin s_2 zD \cdot F_2] \tag{5.2}$$
$$B_1 = - D (s_1 \sin s_1 zD \cdot F_1 + s_2 \sin s_2 zD \cdot F_2)$$

we put the formulas for stresses and displacements in the following form

$$\begin{aligned}
\sigma_x &= - 2G\partial_1\partial_2 \sin s_0 zD \cdot \varphi_0' + A_1 + 2G\partial_2^2 B_1 \\
\sigma_y &= 2G\partial_1\partial_2 \sin s_0 zD \cdot \varphi_0' + A_1 + 2G\partial_1^2 B_1 \qquad (5.3) \\
\sigma_z &= - D^3 [s_1 (\alpha_1 - \beta_1 s_1^2) \sin s_1 zD \cdot F_1 + s_2 (\alpha_1 - \beta_1 s_2^2) \sin s_2 zD \cdot F_2]
\end{aligned}$$

$$\tau_{xy} = G(\partial_1^2 - \partial_2^2)\sin s_0 zD\cdot\varphi_0' - 2G\partial_1\partial_2 B_1$$

$$\tau_{xz} = -G_1 s_0 \partial_2 D\cos s_0 zD\cdot\varphi_0' + \partial_1 D^2\left[(\alpha+\beta s_1^2)\cos s_1 zD\cdot F_1 + (\alpha+\beta s_2^2)\cos s_2 zD\cdot F_2\right] \tag{5.4, 5.5}$$

$$\tau_{yz} = G_1 s_0 \partial_1 D\cos s_0 zD\cdot\varphi_0' + \partial_2 D^2\left[(\alpha+\beta s_1^2)\cos s_1 zD\cdot F_1 + (\alpha+\beta s_2^2)\cos s_2 zD\cdot F_2\right]$$

$$u = -\partial_2 \sin s_0 zD\cdot\varphi_0' - \partial_1 B_1$$

$$v = \partial_1 \sin s_0 zD\cdot\varphi_0' - \partial_2 B_1$$

$$w = D^2\left[(\gamma - \delta s_1^2)\cos s_1 zD\cdot F_1 + (\gamma-\delta s_2^2)\cos s_2 zD\cdot F_2\right]$$

Satisfying the boundary conditions (5.1), we obtain a system of equations for the unknown functions, φ_0', F_1, F_2, which we solve in exactly the same manner as in the case of extension-contraction; as a result, all the quantities will be expressed in terms of the three stress functions, which satisfy the equations

$$Q_1\psi_1 = \frac{t_1}{G_1}, \qquad Q_1\psi_2 = \frac{t_2}{G_1}, \qquad Q_1\psi_3 = \frac{q}{G_1} \tag{5.6}$$

Here

$$Q_1 = D\cos\frac{s_0 hD}{2}\left[(s_1+s_2)\sin(s_1-s_2)\frac{hD}{2} - (s_1-s_2)\sin(s_1+s_2)\frac{hD}{2}\right] \tag{5.7}$$

The final form of the formulas will be

$$\varphi_0' = -\frac{Q_1}{s_0 D^3\cos s_0 hD/2}(\partial_2\psi_1 - \partial_1\psi_2) \tag{5.8}$$

$$F_1 = -\frac{G_1 n}{D^3}\cos\frac{s_0 hD}{2}\left[s_2(\alpha_1 - \beta_1 s_2^2)\sin\frac{s_2 hD}{2}(\partial_1\psi_1 + \partial_2\psi_2) + (\alpha+\beta s_2^2)D\cos\frac{s_2 hD}{2}\cdot\psi_3\right]$$

$$F_2 = \frac{G_1 n}{D^3}\cos\frac{s_0 hD}{2}\left[s_1(\alpha_1 - \beta_1 s_2^2)\sin\frac{s_1 hD}{2}(\partial_1\psi_1 + \partial_2\psi_2) + (\alpha+\beta s_1^2)D\cos\frac{s_1 hD}{2}\cdot\psi_3\right]$$

6. Particular cases of bending. In the problem of bending we also can point out two particular cases which lead to considerable simplifications.

Case 1. Bending by a normal load.

Let $t_1 = t_2$. Then we can set $\psi_1 = \psi_2 = 0$ and introduce a new stress function

$$\psi = \cos\frac{s_0 hD}{2}\cdot\psi_3 \tag{6.1}$$

The function ψ satisfies the equation

$$D\left[(s_1+s_2)\sin(s_1-s_2)\frac{hD}{2}-(s_1-s_2)\sin(s_1+s_2)\frac{hD}{2}\right]\psi=\frac{q}{G_1} \tag{6.2}$$

In this case $\varphi_0{}' = 0$

$$F_1 = -\frac{G_1 n}{D^2}(\alpha+\beta s_2{}^2)\cos\frac{s_2 hD}{2}\psi, \qquad F_2 = \frac{G_1 n}{D^2}(\alpha+\beta s_1{}^2)\cos\frac{s_1 hD}{2}\psi \tag{6.3}$$

Case 2. The surfaces are subjected to tangential forces which have a potential

$$q = 0, \qquad t_1 = \partial_1 t_1, \qquad t_2 = \partial_2 t \tag{6.4}$$

In this case $\varphi_0{}' = 0$

$$\begin{aligned} F_1 &= -\frac{G_1 n}{D} s_2(\alpha_1 - \beta_1 s_2{}^2)\sin\frac{s_2 hD}{2}\psi^* \\ F_2 &= \frac{G_1 n}{D} s_1(\alpha_1 - \beta_1 s_1{}^2)\sin\frac{s_1 hD}{2}\psi^* \end{aligned} \tag{6.5}$$

For the function ψ^* we get the Equation (6.2) in which $q = t$.

7. The homogeneous solutions and the problem of equilibrium of a thick plate of finite dimensions. The solutions of the equations, which are satisfied by the stress functions when the loads on the planes $z = \pm h/2$, are absent, are called the homogeneous solutions. Using the homogeneous solutions one can obtain the stress distribution and displacements in a thick plate of finite dimensions, whose plane surfaces are subjected to an arbitrary load, which can always be broken up into a symmetric and a skew-symmetric part with respect to the middle plane.

Setting in (3.12) and (5.6) $p = \tau_1 = \tau_2 = 0$ and $q = t_1 = t_2 = 0$, we obtain for the functions χ_k and ψ_k the homogeneous linear equations of infinitely high order, corresponding to the form of the operators Q and Q_1. However, if the conditions on the lateral surface (i.e. on the edge) are not satisfied exactly, and if, instead, we require that only the averaged or integral conditions be satisfied, as in the theory of thin plates, then it is sufficient to take for χ_k and ψ_k the particular solutions of the Equations (3.12) and (5.6), namely the biharmonic functions.

The derivation of the formulas for the stresses and displacements, which correspond to the homogeneous solutions, is completely analogous to the derivation for an isotropic layer [1, pp.202-205], and hence we are not going to dwell on it here.

A) A symmetrical stress distribution. Taking the functions χ_k to be biharmonic, we obtain at any point

$$\tau_{xz} = \tau_{yz} = \sigma_z = 0 \tag{7.1}$$

Let us introduce a new biharmonic function $F(x, y)$, setting (7.2)

$$\partial_1\chi_1 = \partial_2\chi_2 = -\frac{F}{2G_1 s_0 (s_1^2 - s_2^2)(1+\nu) h}, \qquad \chi_3 = \frac{(3-\nu) E / G_1 - 8\nu\nu_2}{12E\nu_2 (s_1^2 - s_2^2)(1-\nu)} s_0 F$$

Thus we arrive at the well-known formulas which describe the state of plane stress of a transversely isotropic plate [5]

$$\sigma_x = \partial_2^2 F_1, \qquad \sigma_y = \partial_1^2 F_1, \qquad \tau_{xy} = -\partial_1\partial_2 F_1 \tag{7.3}$$

where

$$F_1 = F + \frac{\nu_2}{2(1+\nu)}\left(\frac{h^2}{12} - z^2\right) D^2 F \tag{7.4}$$

B) A skew-symmetrical stress distribution. We may set $\psi_1 = \psi_2 = 0$ and consider the function ψ to be biharmonic.

Let us introduce a new biharmonic function w_0, related to ψ in the following way

$$\psi = \frac{E}{2G_1 s_1 s_2 (s_1^2 - s_2^2)(1-\nu^2)}\left[w_0 + \frac{h^2}{8(1-\nu)}\left(\frac{2G}{G_1} - \nu_2\right) D^2 w_0\right] \tag{7.5}$$

Then we obtain the well-known relations [5], which are valid for bending

$$u = -z\partial_1 w_1, \qquad v = -z\partial_2 w_1, \qquad w = w_0 + z^2 \frac{\nu_2}{2(1-\nu)} D^2 w_0 \tag{7.6}$$

Here w_0 is the deflection of the middle plane

$$w_1 = w_0 + \frac{E}{2G_1(1-\nu^2)}\left[\frac{h^2}{4} - \left(1 - \frac{G_1\nu_2}{2G}\right)\frac{z^2}{3}\right] D^2 w_0 \tag{7.7}$$

We will not state the formulas for the stresses. In order to solve the problem of equilibrium for a plate of finite dimensions, we should first of all decompose the loading into a symmetric and a skew-symmetric part and determine the corresponding functions χ_k and ψ_k for an infinite layer. After that we find the forces and displacements which occur on the lateral surface, and then superimpose the homogeneous solutions of the types A and B, choosing them in such a manner that the necessary averaged (integral) conditions are satisfied on the lateral surface (on the edge).

In this formulation each of the two problems A and B is reduced to the determination of a function, biharmonic in the domain of the plate.

8. The particular solutions of the equations for the stress functions. Let us consider the cases in which only normal loads p and q are acting on the plane surfaces, and let us look into some special cases of loading. The stress functions χ and ψ satisfy the following equations:

the problem of extension-contraction

$$D\left[(s_1+s_2)\sin(s_1-s_2)\frac{hD}{2}+(s_1-s_2)\sin(s_1+s_2)\frac{hD}{2}\right]\chi=\frac{p}{G_1} \quad (8.1)$$

the problem of bending

$$D\left[(s_1+s_2)\sin(s_1-s_2)\frac{hD}{2}-(s_1-s_2)\sin(s_1+s_2)\frac{hD}{2}\right]\psi=\frac{q}{G_1} \quad (8.2)$$

If the layer is isotropic ($s_1 = s_2 = 1$) then the Equations (8.1) and (8.2) must be replaced by different ones [1, pp.155-157]

$$D^2\left(1+\frac{\sin hD}{hD}\right)\chi=\frac{p}{2G}, \qquad D^2\left(1-\frac{\sin hD}{hD}\right)\psi=\frac{q}{2Gh} \quad (8.3)$$

The question of finding the particular solutions of the equations for the isotropic layer has been treated in sufficient detail in Lur'e's book [1].

In this connection, all the arguments related to the isotropic layer can be applied, without any essential changes, to the case of a transversely isotropic layer, and therefore we can here limit ourselves to outlining the solutions for the two most typical cases of loading.

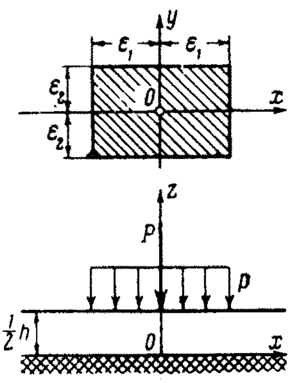

Fig. 3.

1. The loading is a polyharmonic function of the coordinates x, y, i.e. it satisfies the equation

$$D^{2n}p=0 \quad \text{or} \quad D^{2n}q=0 \quad (8.4)$$

In this case χ is an $n+1$ - harmonic function and ψ is an

$n + 2$ - harmonic.

In particular, if p and q are polynomials of the power n with respect to x and y, then χ is a polynomial of the power $n + 2$, and ψ is a polynomial of the power $n + 4$. In looking for χ and ψ in the form of polynomials of respective powers with indeterminate coefficients, we compare the terms on the left- and right-hand sides. Thus, for the coefficients we obtain equations whose number is lower than the number of the unknowns. Consequently, some of the coefficients may be chosen arbitrarily, for instance, they may be set equal to zero.

2. The loading satisfies the equation of free vibrations of a membrane, i.e.

$$D^2 p = -m^2 p \quad \text{or} \quad D^2 q = -m^2 q \qquad (m = \text{const}) \tag{8.5}$$

The functions χ and ψ are defined in the form

$$\chi = -\frac{1}{G_1 m}\,\frac{p}{(s_1+s_2)\sinh[(s_1-s_2)\,mh/2] + (s_1-s_2)\sinh[(s_1+s_2)\,mh/2]} \tag{8.6}$$

$$\psi = -\frac{1}{G_1 m}\,\frac{q}{(s_1+s_2)\sinh[(s_1-s_2)\,mh/2] - (s_1-s_2)\sinh[(s_1+s_2)\,mh/2]} \tag{8.7}$$

Example. An infinite transversely isotropic layer of thickness $h/2$ rests on a smooth, perfectly rigid foundation and is compressed by a normal load, distributed on its upper surface (Fig. 3).

Suppose that the friction on the contact surface is negligibly small, i.e.

$$\tau_{xz} = \tau_{yz} = 0, \qquad w = 0 \ \text{ when } z = 0 \tag{8.8}$$

Then the solution of this problem will be identical to the solution for a layer of thickness h, subjected on both sides to given forces, symmetrical with respect to the middle plane.

Let us assume that the loading can be represented by a Fourier integral

$$p = \int_0^\infty \int_0^\infty Q(\alpha, \beta) \cos \alpha x \cos \beta y \, d\alpha \, d\beta \tag{8.9}$$

where

$$Q = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty p \cos \alpha\xi \cos \beta\eta \, d\xi \, d\eta \tag{8.10}$$

The product $\cos \alpha x \cos \beta y$, obviously, satisfies the Equation (8.5), where $m = \sqrt{(\alpha^2 + \beta^2)}$, and hence χ is given by the formula obtained from (8.6) by performing the summation, i.e. integration

$$\chi = -\frac{1}{G_1}\int_0^\infty\int_0^\infty \frac{Q}{m} \frac{\cos \alpha x \cos \beta y}{(s_1+s_2)\sinh[(s_1-s_2)mh/2] + (s_1-s_2)\sinh[(s_1+s_2)mh/2]}\, d\alpha\, d\beta \quad (8.11)$$

Let, for instance, the compressive loading be uniformly distributed on the area of a rectangle whose sides are $2\varepsilon_1$, $2\varepsilon_2$. Designating the resultant of that loading by P, we have

$$p = -\frac{P}{4\varepsilon_1\varepsilon_2} \quad (8.12)$$

inside the rectangle, and $p = 0$ at other points. Thus we obtain

$$Q = -\frac{P}{\pi^2\varepsilon_1\varepsilon_2\alpha\beta} \sin \alpha\varepsilon_1 \sin \beta\varepsilon_2 \quad (8.13)$$

Letting ε_1 and ε_2 tend to zero (at constant P), we obtain the solution for the case of a concentrated load applied at the origin

$$\chi = \frac{P}{\pi^2 G_1}\int_0^\infty\int_0^\infty \frac{1}{m} \frac{\cos \alpha x \cos \beta y}{(s_1+s_2)\sinh[(s_1-s_2)mh/2] + (s_1-s_2)\sinh[(s_1+s_2)mh/2]}\, d\alpha\, d\beta \quad (8.14)$$

As in the corresponding case of an isotropic layer [1, p.175], the integral (8.14) diverges as m^2, Since the denominator becomes equal to zero for $m = 0$; however, the stresses and displacements, found by means of the function χ from the Formulas (4.3), (3.3), (3.4) and (3.5) are expressed by convergent integrals.

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